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COMMENT

Energy level motion and a supersymmetric Hamiltonian

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Abstract. From the eigenvalue equation $H_\lambda|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle$ where $H_\lambda \equiv H_0 + \lambda V$ one can derive an autonomous system of first-order differential equations for the eigenvalues $E_n(\lambda)$ and the matrix elements $V_{mn}(\lambda) := \langle\psi_m(\lambda)|V|\psi_n(\lambda)\rangle$ where λ is the independent variable. To solve the dynamical system we need the initial values $E_n(\lambda = 0)$ and $|\psi_n(\lambda = 0)\rangle$. Thus one finds the 'motion' of the energy levels $E_n(\lambda)$. We derive the equations of motion for the extended case $H_\lambda = H_0 + \lambda V_1 + \lambda^2 V_2$ and give an application to a supersymmetric Hamiltonian.

Pechukas [1] and Yukawa [2, 3] discussed the 'motion' energy levels $E_n(\lambda)$ where λ plays the role of the time. Let us assume that the eigenfunctions are real orthogonal. Using the orthogonality relation

$$\langle\psi_m(\lambda)|\psi_n(\lambda)\rangle = \delta_{mn} \tag{1}$$

the completeness relation

$$1 = \sum_{n\in I} |\psi_n(\lambda)\rangle\langle\psi_n(\lambda)| \tag{2}$$

$$\left\langle \psi_n(\lambda) \left| \frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}\lambda} \right.\right\rangle = 0 \tag{3}$$

and the assumptions that the eigenvalues are non-degenerate for $\lambda \geqslant 0$, these authors derived the following autonomous system of first-order ordinary differential equations:

$$\mathrm{d}E_n/\mathrm{d}\lambda = p_n \tag{4a}$$

$$\frac{\mathrm{d}p_n}{\mathrm{d}\lambda} = 2 \sum_{m(\neq n)} \frac{V_{mn}V_{nm}}{E_n - E_m} \tag{4b}$$

$$\frac{\mathrm{d}V_{mn}}{\mathrm{d}\lambda} = \sum_{k(\neq m,n)} \left[V_{mk}V_{kn}\left(\frac{1}{E_m - E_k} + \frac{1}{E_n - E_k}\right)\right] - \frac{V_{mn}(p_m - p_n)}{E_m - E_n} \tag{4c}$$

where $p_n(\lambda) := \langle\psi_n(\lambda)|V|\psi_n(\lambda)\rangle$ and $V_{mn}(\lambda) := \langle\psi_m(\lambda)|V|\psi_n(\lambda)\rangle$ $(m \neq n)$. Notice that equation (3) is no longer true if $|\psi_m(\lambda)\rangle$ is complex orthogonal. If we have a finite-dimensional system with N energy levels then the number of differential equations n is given by $n = N + N + \frac{1}{2}N(N-1) \equiv N(\frac{3}{2} + \frac{1}{2}N)$.

Pechukas [1] and Yukawa [2, 3] discussed the dynamical system (4) in connection with quantum chaos (compare [4] and references therein). Moreover, Yukawa [3] showed that the system (4) admits a Lax representation and is completely integrable. Consequently, no chaotic behaviour can be expected for system (4). Nakamura and Lakshmanan [5] gave the equations of motion for the eigenfunctions, namely

$$\frac{\mathrm{d}|\psi_n\rangle}{\mathrm{d}\lambda} = \sum_{m(\neq n)} \frac{V_{mn}}{E_n - E_m} |\psi_m\rangle. \tag{5}$$

Steeb and Van Tonder [6, 7] described the connection with the perturbation theory and considered the extended case $H_\lambda = H_0 + \lambda_1 V_1 + \lambda_2 V_2$. Steeb and Louw [8] discussed energy-dependent constants of motion for system (4). The dependence of the survival probability as well as some thermodynamic quantities (free energy, entropy, specific heat) on λ has been discussed by Steeb [9]. Let us note that Aizu [10] had already described the parameter differentiation of quantum mechanical linear operators 25 years ago. The results given above can be considered as a straightforward application of his results. Furthermore, we note that the system given above is related to the generalised Calogero–Moser system [5, 11].

The question of whether or not energy levels can cross was first discussed by Hund [12]. He studied examples only and conjectured that, in general, no crossing of energy levels can occur. In 1929 von Neumann and Wigner [13] investigated this question more rigorously and found the following theorem. Real symmetric matrices (respectively the Hermitian matrices) with a multiple eigenvalue form a real algebraic variety of codimension 2 (respectively 3) in the space of all real symmetric matrices (respectively all Hermitian matrices). This implies the famous 'non-crossing rule' which asserts that a 'generic' one-parameter family of real symmetric matrices (or two-parameter family of Hermitian matrices) contains no matrix with a multiple eigenvalue. We emphasise that this no-crossing rule has only been proved for finite-dimensional matrices. 'Generic' means that if the Hamiltonian admits symmetries the underlying Hilbert space has to be decomposed into the invariant subspaces.

In some applications we also find different coupling within the interaction. For example, let

$$Q := [b - \lambda(b + b^\dagger)]c^\dagger \tag{6}$$

be the generator of a supersymmetric Hamiltonian [14], where b and c are boson and fermion annihilation operators. Then the Hamiltonian is given by

$$H_\lambda = \{Q, Q^\dagger\} \tag{7}$$

where the braces denote an anticommutator bracket. The Hamiltonian can be written as $H_\lambda = H_0 + V$ where

$$H_0 = c^\dagger c + b^\dagger b \tag{8}$$

and

$$V = -4\lambda c^\dagger c a + 2\lambda a - \lambda a^3 + \lambda^2 a^4 \tag{9}$$

where $a \equiv b^\dagger + b$.

The equations of motion for the Hamiltonian

$$H_\lambda = H_0 + \lambda V_1 + \lambda^2 V_2 \tag{10}$$

can be derived taking into account the assumptions described above. We find

$$\mathrm{d}E_n/\mathrm{d}\lambda = p_n + \lambda q_n \tag{11a}$$

$$\frac{\mathrm{d}p_n}{\mathrm{d}\lambda} = 2 \sum_{m(\neq n)} \frac{V_{1nm}(V_{1mn} + 2\lambda V_{2mn})}{E_n - E_m} \tag{11b}$$

$$\frac{\mathrm{d}q_n}{\mathrm{d}\lambda} = 2 \sum_{m(\neq n)} \frac{V_{2nm}(V_{1mn} + 2\lambda V_{2mn})}{E_n - E_m} \tag{11c}$$

$$\frac{\mathrm{d}V_{1mn}}{\mathrm{d}\lambda} = \sum_{k(\neq m,n)} \frac{(V_{1mk} + 2\lambda V_{2mk})V_{1kn}}{E_m - E_k} + \sum_{k(\neq m,n)} \frac{(V_{1kn} + 2\lambda V_{2kn})V_{1mk}}{E_n - E_k} + \frac{(V_{1mn} + 2\lambda V_{2mn})(p_n - p_m)}{E_m - E_n} \tag{11d}$$

$$\frac{\mathrm{d}V_{2mn}}{\mathrm{d}\lambda} = \sum_{k(\neq m,n)} \frac{(V_{1mk} + 2\lambda V_{2mk})V_{2kn}}{E_m - E_k} + \sum_{k(\neq m,n)} \frac{(V_{1kn} + 2\lambda V_{2kn})V_{2mk}}{E_n - E_k} + \frac{(V_{1mn} + 2\lambda V_{2mn})(q_n - q_m)}{E_m - E_n} \tag{11e}$$

where $p_n(\lambda) := \langle\psi_n(\lambda)|V_1|\psi_n(\lambda)\rangle$ and $q_n(\lambda) := \langle\psi_n(\lambda)|V_2|\psi_n(\lambda)\rangle$.

Equations (11*a*)-(11*e*) cannot be applied to the Hamiltonian (7) since energy levels are degenerate. A basis of the underlying Hilbert space is given by

$$\{|m\rangle|0\rangle;\ |m\rangle c^\dagger|0\rangle;\ m = 0, 1, 2, \ldots\} \tag{12}$$

where

$$|m\rangle := (m!)^{-1/2}(b^\dagger)^m|0\rangle. \tag{13}$$

The matrix representation of the unperturbed Hamiltonian H_0 is given by $H_0 = \mathrm{diag}(0, 1, 1, 2, 2, \ldots)$. In order to apply (11*a*)-(11*d*) we have to decompose the Hilbert space owing to the symmetries. In the present case we find the invariant subspaces $S_1 = \{|m\rangle|0\rangle\}$ and $S_2 = \{|m\rangle c^\dagger|0\rangle\}$. For the subspace with basis S_1 the matrix representation of H_0 is given by $H_0 = \mathrm{diag}(0, 1, 2, \ldots)$ and for the subspace with the basis S_2 we find $H_0 = \mathrm{diag}(1, 2, 3, \ldots)$. In these subspaces we can apply (11*a*)-(11*d*). We have calculated the 'λ evolution' of 100 energy levels of the infinite matrix and taken into account the lowest 10 levels. For the range $0 \leq \lambda \leq 0.5$ no level crossings occur in both subspaces.

Let us assume that the Hamiltonian (10) acts in a finite-dimensional Hilbert space $\mathscr{H}$. The eigenvalues of H_λ satisfy the characteristic equation

$$\det(H_\lambda - E) = 0. \tag{14}$$

This is an algebraic equation of E of degree $N = \dim \mathscr{H}$, with coefficients which are holomorphic in λ. It follows from function theory that the roots of (14) are (branches of) analytic functions in λ with only algebraic singularities.

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